

# EXPANDERS HAVE A SPANNING LIPSCHITZ SUBGRAPH WITH LARGE GIRTH

GÁBOR KUN

**ABSTRACT.** We show that every regular graph with good local expansion admits a spanning Lipschitz subgraph with large girth and minimum degree. We reprove and strengthen the theorem of Gaboriau and Lyons on the dynamical solution to the von Neumann problem. The finite version complements the theorem of Bourgain and Gamburd showing that large girth implies expansion for Cayley graphs of  $SL_2(\mathbb{F}_p)$ . We apply this to the regular case of Thomassen's conjecture stating that every finite graph with large average degree has a subgraph with large girth and average degree.

## 1. INTRODUCTION

The so-called von Neumann problem asked if every non-amenable group has a non-commutative free subgroup. The question did arise in the twenties, when the notion of amenable groups was introduced by von Neumann [27] in order to explain the Banach-Tarski paradox. The original, algebraic version was disproved by Olshanskiy [28]. On the other hand, K. Whyte found a geometric group theoretical solution [32]: a finitely generated group is non-amenable if and only if it has a 4-regular tree as a Lipschitz subgraph. (Benjamini and Schramm [3] proved independently a more general result for arbitrary graphs solving a problem of Deuber, T. Sós and Simonovits [9] (extended by Elek and T. Sós in 2004), and Elek [10] also gave an independent proof.)

Gaboriau and Lyons found a dynamical version in terms of measurable group theory [21]:

**Theorem 1.** *For any finitely generated non-amenable group  $\Gamma$  there is a measurable ergodic essentially free action of  $F_2$  on  $([0, 1]^\Gamma, \nu)$  such that for almost every  $x \in [0, 1]^\Gamma$  the  $F_2$ -orbit of  $x$  is contained by the  $\Gamma$ -orbit of  $x$ .*

This is equivalent to the existence of a free  $F_2$ -action on the Cayley graph of  $\Gamma$  as a factor of IID. Theorem 1 has many different formulations in terms of von Neumann factors, cost and treeings etc. The

---

This research was supported by the “Lendület: Groups and Graphs” Grant.

proof of Gaboriau and Lyons is based on a deep result of Benjamini, Lyons, Peres and Schramm [4] in percolation theory. The main tool of our proof is the Lovász Local Lemma, a standard tool in probabilistic combinatorics. This was realized in the celebrated work of Moser and Tardos by a local algorithm [26].

Theorem 1 gives a way to extend results about groups containing  $F_2$  as a subgroup to every non-amenable group: this was used by Epstein [12], and in Dixmier's unitarizability problem (see Epstein and Monod [13], Monod and Ozawa [18]). In this paper we reprove Theorem 1 and strengthen it with an extra Lipschitz condition on the  $F_2$ -action:

**Theorem 2.** *For any finitely generated non-amenable group  $\Gamma$  there is a measurable ergodic essentially free action of  $F_2$  on  $([0, 1]^\Gamma, \nu)$  such that for almost every  $x \in [0, 1]^\Gamma$  the  $F_2$ -orbit of  $x$  is contained by the  $\Gamma$ -orbit of  $x$ . Moreover, given a Cayley graph  $G$  of  $\Gamma$  there is a constant  $L$  such that  $\text{dist}_G(x, \alpha(x)), \text{dist}_G(x, \beta(x)) < L$  holds for the free generators of the free  $F_2$ -action  $\alpha, \beta$  and almost every  $x \in [0, 1]^\Gamma$ .*

The theorem easily extends to every countable discrete non-amenable group (Gaboriau and Lyons stated it in this generality), but we prefer this formulation, since the Lipschitz condition is more expressive for graphs with finite degrees. The proof will give  $L = O(1/\lambda)$ , where  $\lambda$  is the spectral radius of the random walk. The finite analogue of the theorem is as follows:

**Theorem 3.** *Let  $G$  be a finite  $d$ -regular graph,  $0 < \lambda < 1$  and  $g \in \mathbb{N}$ . Assume that for every  $k < g$  and every vertex  $x \in V(G)$  the number of cycles with length  $k$  containing  $x$  is at most  $(\lambda d)^k$ . Then there is a constant  $L = O(1/\lambda)$  and bijections  $\alpha, \beta : V(G) \rightarrow V(G)$  such that every nontrivial word  $w \in F_2(\alpha, \beta)$  with length  $< g/L$  has no fixed point on  $V(G)$ , and  $\text{dist}(x, \alpha(x)), \text{dist}(x, \beta(x)) < L$  holds for every  $x \in V(G)$ .*

Note that the conditions of the theorem hold for expanders if  $g = O(\log(|V(G)|))$ . Our approach works for arbitrary regular, non-amenable graphs, but it only allows to find an almost regular forest as spanning Lipschitz subgraph (or large girth subgraph) with large minimum degree instead of the free  $F_2$ -action.

**Theorem 4.** *Let  $G$  be a countable  $d$ -regular graph,  $\delta$  a positive integer,  $0 < \lambda < 1$  and  $g \in \mathbb{N} \cup \{\infty\}$ . Assume that for every  $k < g$  and every vertex  $x \in V(G)$  the number of cycles of length  $k$  containing  $x$  is at most  $(\lambda d)^k$ . Then  $G$  has a spanning  $L$ -Lipschitz subgraph  $H$ , where  $L = \max\{2\lceil \frac{\log(12\delta)}{-\log(\lambda)} \rceil + 2; \lceil \frac{\log(\delta)}{\log(d)} \rceil + 1\}$ , with girth at least  $g/L$ , minimum*

degree at least  $\delta$  and maximum degree at most  $(\delta + 1)$ . Moreover, there is a local algorithm that constructs  $H$  almost surely.

This gives already an alternative proof of Theorem 1: if  $g = \infty, \delta = 5$  and  $G$  is the Cayley graph of a non-amenable group then it produces a factor of IID 5-regular forest as a Borel subrelation of a Bernoulli shift  $([0; 1]^\Gamma, \nu)$ . This is ergodic and has cost  $> 2$  so we can use the result of Hjorth [22] to show that it contains an ergodic subrelation generated by an ergodic free action of  $F_2$ .

Our key theorem is the case of graphs with large expansion, when the Lipschitz constant can be 1, so we get actually a spanning subgraph.

**Theorem 5.** *Let  $G$  be a countable  $d$ -regular graph,  $\delta \leq d$  a positive integer and  $g \in \mathbb{N} \cup \{\infty\}$ . Assume that for every  $k < g$  and every vertex  $x \in V(G)$  the number of cycles of length  $k$  containing  $x$  is at most  $(\frac{d}{12\delta})^k$ . Then  $G$  has a spanning subgraph  $H$  with girth at least  $g$  and minimum degree at least  $\delta$ . Moreover, there is a local algorithm that constructs  $H$  almost surely.*

Bourgain and Gamburd [5] proved that Cayley graphs of  $SL_2(\mathbb{F}_p)$  with girth  $\Omega(\log(p))$  (It is generally believed that  $\Omega(1)$  is enough.) are actually expanders: Theorem 4 and 5 complement this theorem.

Theorem 5 gives a strong solution to the regular case of the following conjecture of Thomassen [30]:

**Conjecture 1.** *For every  $d$  and  $g$  there exists a  $D = D(d, g)$  such that every finite graph with average degree at least  $D$  contains a subgraph with average degree at least  $d$  and girth at least  $g$ .*

Thomassen's conjecture is a relaxation of an influential conjecture of Erdős and Hajnal [14, 15] in the seventies, who asked the same for chromatic number instead of average degree. The case of regular graphs is handled by the straight approach of Kühn and Osthus [17], though the general case can not be reduced to this (see Pyber, Rödl and Szemerédi [29]). Kühn and Osthus [17] settled the case  $g = 6$ , while Dellamonica, Koubek, Martin and Rödl [8] proved a directed version of the conjecture. Theorem 5 implies a strengthening of Thomassen's conjecture for regular graphs: we will find a spanning subgraph with the required properties (instead of an arbitrary subgraph).

**Corollary 6.** *Let  $d, D, g$  be positive integers, and  $G$  be a  $D$ -regular graph. Assume that  $D > (12d)^g$ . Then  $G$  has a spanning subgraph with minimal degree at least  $d$  and girth at least  $g$ .*

*Proof.* The number of walks with length  $k$  can be at most  $D^{k-1}$  at any vertex. This is less than  $(\frac{D}{12d})^k$ , hence the condition of Theorem 5 holds. The Corollary follows.  $\square$

**Remark 7.** Note that Theorem 3 immediately allows us to give an alternative dynamical solution to the von Neumann problem for so-called sofic groups introduced by Gromov [20] and Weiss [31]: These groups can be approximated by finite labelled graphs. The ultraproduct of these finite graphs will be a probability space that admits an essentially free action of the group (see Elek, Szegedy [11] for basics on ultraproducts of finite graphs). The orbits of the free  $F_2$ -action provided by Theorem 3 will be contained by the orbits of the group action.

**Future work** In [24] the author proves a measurable version of the Lovász Local Lemma. This allows to extend Theorem 3 to graphings (and arbitrary free actions of a non-amenable group) if  $g < \infty$ . Breuillard and Gelander [6] proved a uniform version of the Tits alternative, showing that for every non-virtually solvable finitely generated group of matrices one can find two elements that are free generators of a free group and are the products of at most  $m$  generators, where  $m$  depends on the dimension only. We hope to reprove this theorem with our methods: the extension of the Lovász Local Lemma in [24] may be the first step in this direction.

## 2. DEFINITIONS

We say that a graph is  $d$ -regular if every vertex has degree  $d$ . The girth of a graph  $G$  denoted by  $g(G)$  is the length of the shortest cycle. An acyclic graph is called a forest, a connected forest is called a tree. The minimum degree of  $G$  is denoted by  $\delta(G)$ . A matching is a set of edges that covers every vertex at most once. The matching is perfect if it covers every vertex exactly once. Given a non-perfect matching an *alternating path* is a path with an odd number of edges such that every other edge in the path is in the matching, and the two endpoints of the path are unmatched. Switching the matching and non-matching edges of an alternating path increases the size of the matching by one: this is the standard way to find a perfect matching.

**Definition 8.** We say that the graph  $H$  is an  $L$ -Lipschitz subgraph of the graph  $G$  if  $V(H) \subseteq V(G)$ , and for every edge  $(xy) \in E(H)$  the distance of  $x$  and  $y$  is at most  $L$  in the graph  $G$ . We say that  $H$  is a spanning  $L$ -Lipschitz subgraph of  $H$  if it is an  $L$ -Lipschitz subgraph and  $V(H) = V(G)$ .

Note that the (spanning) 1-Lipschitz subgraphs of a graph are exactly the (spanning) subgraphs. The Cayley graph of the group  $\Gamma$  generated by  $S \subseteq \Gamma$  is a graph  $G$ , where  $V(G) = \Gamma$  and  $E(G) = \{(x, y) : x, y \in \Gamma, x^{-1}y \in S\}$ . We will denote this  $S$ -colored graph by  $\text{Cay}(\Gamma, S)$ . We will assume that  $S = S^{-1}$ , so the graph will be undirected. We will sometimes consider the (directed) labeling of the vertices by the elements of  $S$ .

We will work with so called *graphings*, these are graphs on a standard Borel measure space with a Borel edge set: We will only consider graph(ing)s with bounded degree on a probability measure space. The normalized spectral radius of (the self-adjoint operator corresponding to) a graphing will be denoted by  $\rho$ . A *treeing* is a graphing that is a forest (as a graph).

We say - following Kesten - that a (finitely generated) group is non-amenable if given its  $d$ -regular Cayley graph there is a  $\lambda < 1$  such that the number of  $k$ -walks at any vertex is at most  $(\lambda d)^k$ . (The existence of  $\lambda$  is independent of the choice of the Cayley graph.) Given a measure  $\nu$  on the interval  $[0; 1]$  and a group  $G$  we will consider the Bernoulli shift  $[0; 1]^G$  with the product measure and the measure-preserving, essentially free  $G$ -action on it.

The notion of local (randomized) algorithms will play an important role for us. Consider the space  $\mathcal{G}_b$  of connected, rooted graphs with maximum degree at most  $b$ . Set  $\mathcal{F}_b = \{(G, f) : G \in \mathcal{G}_b, f \in [0; 1]^{V(G)}\}$ . Consider the  $\sigma$ -algebra generated by the following sets: given a finite, connected, rooted graph  $G \in \mathcal{G}_b$  and  $B \subseteq [0; 1]^{V(G)}$  Borel consider the set  $\{(H, f) \in \mathcal{F}_b : \exists r \text{ s.t. the rooted } r\text{-ball of } H \text{ is isomorphic to } G \text{ and the restriction of } h \text{ to the ball equals } f\}$ . Let us call the elements of the  $\sigma$ -algebra generated by these sets Borel. Given a finite set  $C$  we call a mapping  $\varphi : \mathcal{G}_b \rightarrow C$  Borel if the pre-image of any element of  $C$  is Borel. Given a degree bound  $b$ , a finite set  $C$  and a measurable mapping  $\varphi : \mathcal{G}_b \rightarrow C$  we call the following randomized algorithm a **local algorithm**: Given a graph  $G$  we generate a random  $g \in [0; 1]^{V(G)}$  by generating a random number in  $[0; 1]$  at every vertex uniformly and independently. Then the algorithm assigns to every vertex  $v$  of  $G$  the value  $f(H, h) \in C$ , where  $H$  is the rooted graph isomorphic to the connected component of  $v$  in  $G$  rooted at  $v$ , and  $h$  is the restriction of  $g$  to  $H$ .

### 3. THE PROOF OF THEOREM 5

Consider the following probability distribution on the subsets of  $E(G)$ : choose  $\delta$  distinct edges at every vertex independently, uniformly

at random, and let  $E(H)$  consist of these edges. We will use the Lovász Local Lemma to prove that  $H$  can satisfy the conditions of the theorem: in case of finite graphs this will happen with positive probability. The Lovász Local Lemma was originally proved by Erdős and Lovász [16] (see [1] for background and applications). However, we will need a new version of Moser and Tardos [26] that can be realized by a randomized local algorithm. We will consider a set of mutually independent random variables. Given an event  $A$  determined by these variables we will denote by  $vbl(A)$  the unique minimal set of variables that determine the event  $A$ : such a set clearly exists.

**Lemma 9.** [26] *Let  $\mathcal{V}$  be a set of mutually independent random variables in a probability space. Let  $\mathcal{A}$  be a set of events determined by these variables. If there exists an assignment  $x : \mathcal{A} \rightarrow (0; 1)$  such that*

$$\forall A \in \mathcal{A} \Pr[A] \leq x(A) \prod_{vbl(A) \cap vbl(B) \neq \emptyset} (1 - x(B))$$

*then there exists an assignment of all variables in  $\mathcal{V}$  violating any of the events in  $\mathcal{A}$ . There is a randomized local algorithm that finds an evaluation such that any of the events will be almost surely violated.*

Moser and Tardos had a more precise theorem but for finitely many events: their proof works in this generality, too. The set of variables  $\mathcal{V}$  will correspond to the vertices of  $G$ . We will call a cycle *short* if it is shorter than  $g$ . The "bad events" of  $\mathcal{A}$  correspond to short cycles: for every short cycle  $C$  consider the bad event that  $H$  contains this cycle. We will write " $C - C'$ " to indicate that  $vbl(C) \cap vbl(C') \neq \emptyset$ .

**Claim:** Let  $x_1, \dots, x_k$  be a cycle in  $G$ . Then

$$\Pr((x_i, x_{i+1}) \in E(H) \text{ for } i = 1, \dots, k) \leq \left(\frac{2\delta}{d}\right)^k.$$

*Proof.* We suffice to show that for every  $i$  the conditional probability  $\Pr((x_i, x_{i+1}) \in E(H) | (x_1, x_2), \dots, (x_{i-1}, x_i) \in E(H))$  is at most  $\Pr((x_i, x_{i+1}) \in E(H)) = 2\frac{\delta}{d} - \frac{\delta^2}{d^2} < \frac{2\delta}{d}$ . We will prove the following, equivalent inequality:

$$\Pr((x_1, x_2), \dots, (x_{i-1}, x_i) \in E(H) | (x_i, x_{i+1}) \notin E(H)) \geq \Pr((x_1, x_2), \dots, (x_{i-1}, x_i) \in E(H)).$$

Consider the following distribution on the subsets of  $E(G) \setminus (x_i, x_{i+1})$ : choose  $\delta$  edges at every vertex independently, and let  $L$  be the union of these edges. The probability that the edges  $(x_1, x_2), \dots, (x_{i-1}, x_i)$  are in  $L$  equals to the left hand side, while the probability that  $E(H)$  contains these edges is on the right hand side. The Claim follows.  $\square$

Given a short cycle  $C$  in  $G$  let  $A_C$  denote the event that  $E(H)$  contains the edges of  $C$ . Set  $x(A_C) = (\frac{3\delta}{d})^k$ , where  $k$  is the length of  $C$ . According to the Claim we suffice to show for every short cycle  $C$  that

$$(\frac{2\delta}{d})^k \leq x(A_C)\Pi_{C-C'}(1 - x(A_{C'})),$$

what is the upper bound required by the Local Lemma. We use the bound on the number of cycles sharing a vertex:

$$\Pi_{C-C'}(1 - x_{C'}) \geq \Pi_{1 \leq i < g}^k (1 - (3\delta/d)^i)^{(\frac{d}{12\delta})^i}.$$

On the other hand,

$$\begin{aligned} \Pi_{1 \leq i < g} (1 - (\frac{3\delta}{d})^i)^{(\frac{d}{12\delta})^i} &= \exp\left(\sum_{1 \leq i < g} (\frac{d}{12\delta})^i \log(1 - (\frac{3\delta}{d})^i)\right) \geq \\ \exp\left(\log\left(1 - \sum_{1 \leq i < g} (\frac{d}{12\delta})^i (\frac{3\delta}{d})^i\right)\right) &= 1 - \sum_{1 \leq i < g} (\frac{3\delta}{d})^i (\frac{d}{12\delta})^i \geq \\ 1 - \sum_{i=1}^{\infty} (\frac{3\delta}{d})^i (\frac{d}{12\delta})^i &= 1 - \sum_{i=1}^{\infty} 4^{-i} = 1 - 1/3 = 2/3. \end{aligned}$$

The first inequality holds, since  $f(x) = \frac{\log(1-x)}{x}$  is monotone decreasing on the interval  $(0; 1)$ , and  $\sum_{i=1}^{\infty} 4^{-i} < 1$ . Hence

$$\begin{aligned} (\frac{2\delta}{d})^k &= (\frac{3\delta}{d})^k (\frac{2}{3})^k \leq x(A_C) \Pi_{1 \leq i < g}^k (1 - (\frac{3\delta}{d})^i)^{(\frac{d}{12\delta})^i} \leq \\ x(A_C) \Pi_{C-C'}(1 - x(A_{C'})). \end{aligned}$$

This completes the proof of the theorem.

#### 4. THE PROOF OF THEOREM 4

Consider the following (power) graph  $G^{(L/2)}$ :  $V(G^{(L/2)}) = V(G)$ , and the multiplicity of the edge  $(x, y)$  is the number of walks with length  $L/2$  from  $x$  to  $y$ . The graph  $G$  is  $d^{L/2}$ -regular. The number of walks with length  $k < 2g/L$  is at most  $(\lambda d)^{kL/2} < (\frac{d^{L/2}}{12\delta})^k$  at every vertex. This is an upper bound on the number of cycles, too, so we can apply Theorem 5 in order to get a spanning subgraph  $H'$  of  $G^{(L/2)}$  with minimum degree  $\geq \delta$  and girth  $> 2g/L$ . This graph  $H'$  will be a spanning  $L/2$ -Lipshitz subgraph of  $V(G)$ . We will use the following lemma in order to get an almost regular Lipschitz subgraph.

**Lemma 10.** *Let  $G$  be a countable, loopless, undirected graph with bounded maximum degree and minimum degree at least  $\delta \in \mathbb{N}$ . Then  $G$  has a spanning 2-Lipschitz subgraph  $H$  with girth at least  $\frac{g(G)}{2}$ , minimum degree at least  $\delta$  and maximum degree at most  $(\delta + 1)$ . Moreover, there is a local algorithm that produces  $H$  almost surely.*

*Proof.* First we find a spanning subgraph  $G_1$  of  $G$  such that  $\delta(G_1) \geq \delta$  and  $G_1$  has no distinct, adjacent pair of vertices with degree  $> \delta$ : If there is an edge connecting vertices with degree  $> \delta$  then we remove this edge. We iterate this process until we get the desired subgraph  $G_1$ .

Next we will find a spanning 2-Lipschitz subgraph  $H$  of  $G_1$  s.t.  $\delta(H) = \delta, \Delta(H) \leq \delta + 1$  and the degree of every vertex is at most its degree in  $H$ : For every vertex  $x$  of  $G_1$  of degree  $> \delta$  let  $v_{x,1}, \dots, v_{x,deg(x)}$  denote the neighbors of  $x$ , and set

$$E(G_2) = \{(v_{x,2i-1}, v_{x,2i}) : deg(x) > \delta, 1 \leq i \leq \lfloor \frac{deg(x)-\delta}{2} \rfloor\} \cup E(G_1) \setminus \{(v, v_{x,i}) : deg(x) > \delta, 1 \leq i \leq 2\lfloor \frac{deg(x)-\delta}{2} \rfloor\}. \quad \square$$

## 5. THE PROOF OF THEOREM 3

**Lemma 11.** *Let  $\delta \geq 4$  an even integer and  $G$  a finite graph with minimum degree  $\delta$  and maximum degree at most  $(\delta + 1)$ . Assume that  $G$  has no adjacent pair of vertices with degree  $(\delta + 1)$ . Then  $G$  has a  $\delta$ -regular spanning 3-Lipschitz subgraph  $H$  with girth at least  $g(G)/3$ .*

*Proof.* Call the vertices with degree  $(\delta + 1)$  *special*. We will remove the cycles of  $G$  iteratively in order to end up at a forest as spanning subgraph. Special vertices will still have odd degree, and the other vertices will have even degree. We will do the following surgery for a well chosen path  $x_1, \dots, x_{k-1}$  connecting special vertices: Add an extra vertex to both ends of every path so we get a path  $x_0, \dots, x_k$ , where  $x_1$  and  $x_{k-1}$  are special vertices. Remove all edges of the path and add edges of the form  $(x_i, x_{i+2})$ , where  $i = 0, \dots, (k-2)$ . The degree of the special vertices,  $x_1$  and  $x_{k-1}$  has decreased by one. The degree of the other vertices has not changed. We can do this surgery for many paths paralelly if these are edge-disjoint (including the additional edges).

There are vertices of degree one connected by a path in the forest that has at most one vertex with degree  $> 2$  (in particular at most one special vertex) in its interior. We do the surgery for every such path: we can choose the extra edge at the endpoint so that our paths will be edge-disjoint. We iterate the process for the remaining forest until we match all special vertices and do the surgery.

We claim that we will have a 3-Lipschitz subgraph of  $G$  in the end. The only danger is that edges get longer and longer in the iteration. But the edges we use are only the edges of the forest plus the edges added at the ends of the paths. The only way we can reuse a 2-edge of an intermediate Lipschitz subgraph is that one of its endpoints is a special vertex we use later (as the endpoint of a path surged). But this can happen to every 2-edge at most once, while the other edge it will be joined with should be an original edge of  $G$ .  $\square$

Theorem 4 and the lemma give a 4-regular spanning Lipschitz subgraph with large girth. This can be partitioned into two 2-regular spanning subgraphs. The edges of these 2-regular graphs have an Eulerian



orientation, so the two digraphs could be actually the graphs of the functions  $\alpha$  and  $\beta$ , respectively. This completes the proof of Theorem 3.

## 6. THE PROOF OF THEOREM 2

Theorem 4 gives an almost regular tree as a Lipschitz subgraph. Next we will find a regular spanning Lipschitz subtree of our almost regular tree. In the regular case we will be able to find a 2-regular subtree with an Eulerian orientation of the edges: such a digraph can be the graph of the desired function  $\alpha$  ( $\beta$ ). All these steps of the proof are essentially matching problems after a reformulation. We can use the fact that our treeings have good expansion properties to find short alternating paths and to get a local algorithm. This strategy was used by Lyons and Nazarov [25], who proved that the Cayley graph of a countable, non-amenable group admits a factor of IID matching. (See also the work of Abért, Csikvári, Frenkel and the author [2], and Csóka and Lippner [7].)

**Lemma 12.** *Let  $\delta \geq 3$  an odd integer and  $G$  a treeing with minimum degree  $\delta$  and maximum degree at most  $(\delta + 1)$ . Assume that  $G$  has no adjacent pair of vertices with degree  $(\delta + 1)$ . Then  $G$  has a spanning 2-Lipschitz subgraph that is an a.e.  $\delta$ -regular treeing, moreover, it can be constructed by a local algorithm.*

*Proof.* Call the vertices with degree  $(\delta + 1)$  *special*. First we will find an edge-disjoint set of paths connecting special vertices such that every special vertex is the endpoint of exactly one path. Then we will make the following surgery. Add an extra vertex to both ends of every path: these new paths can be still edge-disjoint, since special vertices have odd degree. For every new path  $x_0, \dots, x_k$  remove all edges of the path and add edges of the form  $(x_i, x_{i+2})$ , where  $i = 0, \dots, (k - 2)$ . The degree of the special vertices,  $x_1$  and  $x_{k-1}$  has decreased by one. The degree of the other vertices has not changed, hence  $H$  is  $\delta$ -regular.

Altogether, we suffice to find an edge-disjoint set of paths connecting special vertices such that every special vertex is the endpoint of exactly one path. We will proceed with the following local algorithm: Assume that we have already a set of edge-disjoint paths matching all special vertices, but a set of measure  $\varepsilon$ . The  $r$ -ball centered at any vertex of  $G$  has volume greater than  $\delta^r$ , hence there will be unmatched vertices at distance at most  $2[\log(1/\varepsilon)/\log(\delta)] + 2$ . Pick a maximal set of edge-disjoint paths connecting special vertices s.t. the set of endpoints has positive measure. Add these paths to our set of paths and remove the possible double edges: this will be still a set of paths connecting pairs of

special vertices. Continue the process until almost every special vertex will be matched. The total length of the paths we have worked with is at most  $\int_0^1 2[\log(1/\varepsilon)/\log(\delta)] + 2 d\varepsilon < \infty$ , hence the limit of this process is a.e. defined by the Borel-Cantelli lemma (i.e. the position of a.e. edge stabilizes).  $\square$

Note that we only used the fact that the volume of an  $r$ -ball is barely larger than a linear function of  $r$ . The proof of the lemma would work for many other graphings, not only treeings.

This lemma finds a regular Lipschitz subtreeing of the almost regular Lipschitz subtreeing of the Bernoulli shift provided by Theorem 4. This is a factor of IID Lipschitz subgraph of the original Cayley graph. It has a matching  $\mathcal{M}$  by the result of Lyons and Nazarov [25].

We will orient the edges of  $\mathcal{M}$  randomly using the Local Lemma in order to get a partition into two parts with the same measure s.t. every vertex has at least  $\frac{2}{5}$  portion of its neighbors in the other part. Then we will find a matching in the rest of the graph s.t. every edge connects an endpoint of an edge in  $\mathcal{M}$  to a starting point of an edge in  $\mathcal{M}$ . If we found this we could orient these edges to get a 2-regular graph with an Eulerian orientation. Such a digraph could be the graph of the function  $\alpha(\beta)$ .

**Lemma 13.** *Let  $G$  be a  $(d+1)$ -regular graphing, where  $d > 100$  and  $\mathcal{M}$  a matching of  $G$ . Then there is a local algorithm that orients the edges of  $\mathcal{M}$  such that for the induced partition of  $V(G)$  into in- and out-vertices the following holds: a.e.  $v \in V(G)$  has at least  $\frac{2d}{5}$  of its  $d$  non-matching neighbors in the other class of the partition than  $v$ .*

*Proof.* Consider the independent, uniform, random orientation of the edges. We will apply the Lovász Local Lemma to this probability distribution. The Chernoff inequality implies that the probability that the neighbors of a vertex are badly directed is at most  $e^{-d^2/100}$ . We choose for every bad event  $A_v$  (corresponding to a vertex  $v$ )  $x = x(A_v) = 1/(d+1)$ . We only need to check the condition of the Local Lemma:  $(1-x)^d x > \frac{1}{e(d+1)} > e^{-d^2/200}$ , where the second inequality uses that  $d > 100$ .  $\square$

**Lemma 14.** *Let  $G$  be a  $d$ -regular expander graphing and  $\rho > 0$  its normalized spectral radius. Partition  $V(G)$  into two disjoint sets  $A$  and  $B$  with equal measure such that for a. e.  $x \in V(G)$  at least  $\frac{2d}{5}$  neighbors of  $x$  will be in the other set of the partition. If  $50\rho < 1$  then the bipartite graphing will be an expander: for every  $S \subseteq A$  (or  $B$ ) measurable with  $|S| < |A|/2$  we have  $|N(S)| > \frac{3|S|}{2}$ .*

*Proof.* Let  $S \subset A$  measurable,  $|S| \leq \frac{|A|}{2}$ . The Expander Mixing Lemma implies  $E(S, N(S)) \leq \frac{d|S||N(S)|}{|A|+|B|} + \rho d \sqrt{|S||N(S)|}$ . On the other hand, every vertex of  $S$  has at least  $2d/5$  neighbors in  $B$ , i.e. in  $N(S)$ , hence  $2d|S|/5 \leq \frac{d|S||N(S)|}{|A|+|B|} + \rho d \sqrt{|S||N(S)|}$ . Altogether,  $\frac{4}{5} \leq \frac{|N(S)|}{|B|} + 2\rho \sqrt{\frac{|N(S)|}{|S|}}$ : the lemma follows, since  $50\rho < 1$ .  $\square$

We are ready to finish our proof: assume that we got a  $d$ -regular forest, where  $d > 10000$ , as a factor of IID spanning Lipschitz subgraph. This has a factor of IID matching and an orientation of the matching edges s.t. it induces a partition of the vertices of the Bernoulli shift into two equal parts, and every vertex has at least  $\frac{2}{5}$  portion of its neighbors in the other half of the partition. We will show that there is a factor of IID matching of this bipartite graphing: since we have started at an at least 10000-regular treeing, we had the Ramanujan spectra, hence  $50\rho = \frac{100\sqrt{d-1}}{d} < 1$ , since  $d > 10000$ . The lemma implies that the bipartite graphing is an expander, so we have short alternating paths and a local algorithm to find a matching. This completes the proof of Theorem 2.

**Acknowledgement.** The author is strongly indebted to Gábor Elek who asked him whether Theorem 3 holds and introduced him to this subject. The author thanks to Miklós Abért, László Pyber and Péter Varjú for their helpful remarks.

## REFERENCES

- [1] Noga Alon and Joel H. Spencer, The probabilistic method, Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience [John Wiley and Sons], New York, second edition, 2000. With an appendix on the life and work of Paul Erdős.
- [2] M. Abért, P. Csikvári, P. Frenkel, G. Kun, Matchings in Benjamini-Schramm convergent graph sequences, manuscript,
- [3] I. Benjamini and O. Schramm, Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant, *Geom. Funct. Anal.* Vol. 7 No. 3, 403–419 (1997),
- [4] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, Uniform spanning forests, *Ann. Probab.*, 29(1):1–65, 2001,
- [5] J. Bourgain and A. Gamburd, Uniform expansion bounds for Cayley graphs of  $SL_2(\mathbb{F}_p)$ , *Annals of Mathematics*, 167 (2008), 625–642,
- [6] E. Breuillard and T. Gelander, Uniform independence in linear groups, *Inventiones mathematicae* (2008), Volume 173, Issue 2, pp 225–263,
- [7] E. Csóka, G. Lippner, Perfect matchings in expander graphings, manuscript,
- [8] D. Dellamonica, V. Koubek, D. Martin and V. Rödl, On a conjecture of Thomassen, *J. Graph Theory*, 2011, vol. 67, no. 4, pg. 316 – 331,

- [9] W. A. Deuber, M. Simonovits and V. T. Sós, A note on paradoxical metric spaces, *Studia Scientiarum Mathematicarum Hungarica* (1995), 17–23, see <http://www.renyi.hu/~miki/walter07.pdf> for an extended version by G. Elek and T. Sós,
- [10] G. Elek, Amenability,  $l_p$ -homologies and translation invariant functionals, *Journal of the Australian Mathematical Society* 65 (1), (1998) 111–119,
- [11] G. Elek, B. Szegedy, A measure-theory approach to the theory of dense hypergraphs, *Advances in Mathematics* 231 (2012) 1731–1772,
- [12] I. Epstein, Orbit inequivalent actions of non-amenable groups, preprint,
- [13] I. Epstein and N. Monod, Nonunitarizable representations and random forests, *Int. Math. Res. Not. IMRN*, (22):4336–4353, 2009.
- [14] P. Erdős, Problems and results in chromatic graph theory. In *Proof Techniques in Graph Theory* (Proc. Second Ann Arbor Graph Theory Conf., Ann Arbor, Mich., 1968), pages 2735. Academic Press, New York, 1969. 1,
- [15] P. Erdős, Some unsolved problems in graph theory and combinatorial analysis. In *Combinatorial Mathematics and its Applications* (Proc. Conf., Oxford, 1969), pages 97109. Academic Press, London, 1971. 1,
- [16] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions. In *Infinite and finite sets* (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II, pages 609627. *Colloq. Math. Soc. Janos Bolyai*, Vol. 10. North-Holland, Amsterdam, 1975. 2
- [17] D. Kühn and D. Osthus, "Every graph of sufficiently large average degree contains a  $C_4$ -free subgraph of large average degree", *Combinatorica*, 24(1): 155162, 2004,
- [18] N. Monod and N. Ozawa, The Dixmier problem, lamplighters and Burnside groups. *J. Funct. Anal.*, 258(1):255–259, 2010.
- [19] D. Gaboriau, "Cost des relations d'équivalence et des groupes", (in French) [Cost of equivalence relations and of groups], *Inventiones Mathematicae*, 139 (2000), no. 1, 41–98,
- [20] M. Gromov, Endomorphisms of symbolic algebraic varieties, *J. Eur. Math. Soc.* 1 (1999) no. 2, 109–197.
- [21] D. Gaboriau, R. Lyons, "A Measurable-Group-Theoretic Solution to von Neumann's Problem", *Inventiones Mathematicae*, 177 (2009), 533–540,
- [22] Greg Hjorth, A lemma for cost attained. *Ann. Pure Appl. Logic*, 143(1-3):87–102, 2006,
- [23] A. Kechris, Global aspects of ergodic group actions, *Mathematical Surveys and Monographs*, 160. American Mathematical Society, 2010
- [24] G. Kun, A measurable version of the Lovász local Lemma, manuscript,
- [25] R. Lyons, F. Nazarov, Perfect Matchings as IID Factors on Non-Amenable Groups, *European Journal of Combinatorics*, Volume 32 Issue 7, (2011), Pages 1115–1125,
- [26] R. Moser, G. Tardos, "A constructive proof of the general Lovasz Local Lemma", *Journal of the ACM* 57 (2010) (2) Art. 11,
- [27] J. von Neumann, Zur allgemeinen theorie des Masses. *Fund. Math.*, 13:73–116, 1929.
- [28] Alexander Ju Olshanskiy, On the question of the existence of an invariant mean on a group, *Uspekhi Mat. Nauk*, 35(4(214)):199?200, 1980,

- [29] László Pyber, Vojtech Rödl and Endre Szemerédi, Dense graphs without 3-regular subgraphs, J. Combin. Theory Ser. B, 63(1):41–54, 1995. 1, 6
- [30] Carsten Thomassen, Girth in graphs, J. Combin. Theory Ser. B, 35(2):129–141, 1983.1
- [31] B. Weiss, Sofic groups and dynamical systems (Ergodic theory and harmonic analysis, Mumbai, 1999) Sankhya Ser. A. 62 (2000) no. 3, 350-359,
- [32] K. Whyte, Amenability, bi-Lipschitz equivalence, and the von Neumann conjecture. Duke Math. J., 99(1):93–112, 1999.

*E-mail address:* kungabor@cs.elte.hu

EÖTVÖS UNIVERSITY, BUDAPEST